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# SQUARE FUNCTION INEQUALITIES FOR NON-COMMUTATIVE MARTINGALES

BY

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ABSTRACT

We prove a non-commutative version of the weak-type (1,1) boundedness of square functions of martingales. More precisely, we prove that there is an absolute constant $K$ with the following property: if $\mathcal{M}$ is a semi-finite von Neumann algebra with a faithful normal trace $\tau$ and $(\mathcal{M}_n)_{n=1}^{\infty}$ is an increasing filtration of von Neumann subalgebras of $\mathcal{M}$ then for any martingale $x = (x_n)_{n=1}^{\infty}$ in $L^1(\mathcal{M}, \tau)$, adapted to $(\mathcal{M}_n)_{n=1}^{\infty}$, there is a decomposition into two sequences $(x_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ with $x_n = y_n + z_n$ for every $n \geq 1$ and such that

$$\left\| \left( \sum_{n=1}^{\infty} |dy_n|^2 \right)^{1/2} \right\|_{1,\infty} + \left\| \left( \sum_{n=1}^{\infty} |dz_n^*|^2 \right)^{1/2} \right\|_{1,\infty} \leq K\|x\|_1.$$

This generalizes a result of Burkholder from classical martingale theory to non-commutative martingales. We also include some applications to martingale Hardy spaces.

## 0. Introduction

The starting point of this paper is a classical result of Burkholder on weak-type (1,1) boundedness of square functions of martingales, which we state explicitly:

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THEOREM 0.1 ([B1]): *Let $(f_n)_{n=1}^{\infty}$ be a martingale on a probability space $(\Omega, \Sigma, P)$ and $S(f) = (\sum_{n=1}^{\infty} |f_n - f_{n-1}|^2)^{1/2}$. Then there exists an absolute constant $M$ such that for every $\lambda > 0$,*

$$\lambda P(S(f) > \lambda) \leq M \sup_n \mathbb{E}(|f_n|).$$

The quantity $S(f)$ is called the square function of the martingale $(f_n)_{n=1}^{\infty}$ and inequalities of the likes of the one in Theorem 0.1 are referred to as square function inequalities. Square function inequalities have been a very useful tool in various parts of analysis and have been extended to cover many varieties of differing contexts (see for instance the book [EG] for extensions and applications to Harmonic analysis, the survey article [B2] for their use in Banach space theory). Note that martingale square functions are closely related to martingale transforms. In fact, Theorem 0.1 can be deduced from the weak-type (1,1) boundedness of martingale transforms via Khintchine inequalities or by Doob's identity (see for instance [G, Chap. II]).

As parts of the general development of quantum probability theory (see the books by Meyer [M] and Parthasarathy [P] for more information on quantum probability along with its connections with other fields of mathematics such as mathematical physics and classical probability), the theory of non-commutative martingales has received considerable attention in recent years. Indeed, many of the classical inequalities in the usual (commutative) martingale theory have been generalized to the non-commutative setting. Let us recall some sample contributions by several authors. For instance, pointwise convergence of non-commutative martingales was considered in [C] and [DN]. The non-commutative Burkholder–Gundy inequalities and a non-commutative analogue of Stein's inequality were proved by Pisier and Xu in [PX]. A non-commutative analogue of Doob's maximal inequality was successfully formulated and proved by Junge in [J] and non-commutative Burkholder/Rosenthal inequalities were studied by Junge and Xu in [JX1] among many other related topics. These different results pave the way to the consideration of non-commutative martingale Hardy spaces and non-commutative martingale $BMO$ which are very closely related to square functions.

It is a natural question to consider if Theorem 0.1 has non-commutative counterparts. We recall that non-commutative martingale transforms are of weak-type (1,1) ([R1]). However, unlike the classical case, a non-commutative analogue of Theorem 0.1 cannot be deduced directly from the weak-type (1,1) boundedness of martingale transforms as (at least at the time of this writing) there is no adequate Khintchine inequality for non-commutative weak-$L^1$-spaces.

Our main result is Theorem 2.1 in Section 2 below which is an analogue of Theorem 0.1 for non-commutative settings. One should note, however, that for non-commutative Burkholder–Gundy inequalities obtained in [PX], the square functions were formulated differently according to $p \geq 2$ or $1 \leq p < 2$ (this phenomenon was first discovered for non-commutative Khintchine inequalities, [LP, LPP]) so one is lead to consider the appropriate form of square functions for the non-commutative weak-$L^1$-spaces which, as expected, should be similar to the case $1 \leq p < 2$ previously studied for the Burkholder–Gundy inequalities.

The paper is for the most part self-contained. Our method of proof depends heavily on a non-commutative version of the classical Doob's maximal inequality obtained by Cuculescu [C]. The novelty of our approach is the use of triangular truncations to decompose any given martingale into a sum of sequences whose (appropriate) square functions belong to the corresponding non-commutative weak-$L^1$-space.

The paper is organized as follows: In Section 1 below, we set some basic preliminary background concerning non-commutative spaces and non-commutative martingale theory that will be needed throughout. Section 2 is devoted mainly to the statement and proof of the main result. We end the paper with some consequences of the main result to martingale Hardy spaces and martingale $BMO$ spaces.

Our notation and terminology are standard and may be found in the books [KR1], [KR2] and [T].

## 1. Preliminaries

Throughout this paper, $\mathcal{M}$ is a semi-finite von Neumann algebra with a normal faithful semi-finite trace $\tau$. The identity element of $\mathcal{M}$ is denoted by $\mathbf{1}$. For $0 < p \leq \infty$, let $L^p(\mathcal{M}, \tau)$ be the associated non-commutative $L^p$-space (see for instance [DI] and [N]). Note that if $p = \infty$, $L^\infty(\mathcal{M}, \tau)$ is just $\mathcal{M}$ with the usual operator norm; also recall that for $0 < p < \infty$, the (quasi)-norm on $L^p(\mathcal{M}, \tau)$ is defined by

$$\|x\|_p = (\tau(|x|^p))^{1/p}, \quad x \in L^p(\mathcal{M}, \tau),$$

where $|x| = (x^*x)^{1/2}$ is the usual modulus of $x$.

Let us recall the general setup for martingales. The reader is referred to [DO] and [G] for the classical (commutative) martingale theory. Let $(\mathcal{M}_n)_{n=1}^\infty$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that the union of

$\mathcal{M}_n$'s is weak*-dense in $\mathcal{M}$. For each $n \geq 1$, assume that there is a normal conditional expectation $\mathcal{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$ satisfying:

(i) $\mathcal{E}_n(axb) = a\mathcal{E}_n(x)b$ for all $a, b \in \mathcal{M}_n$ and $x \in \mathcal{M}$;

(ii) $\tau \circ \mathcal{E}_n = \tau$.

It is clear that for every $m$ and $n$ in $\mathbb{N}$, $\mathcal{E}_m\mathcal{E}_n = \mathcal{E}_n\mathcal{E}_m = \mathcal{E}_{\min(n,m)}$. Since $\mathcal{E}_n$ is trace preserving, it extends as a contractive projection from $L^p(\mathcal{M}, \tau)$ onto $L^p(\mathcal{M}_n, \tau_n)$ for all $1 \leq p \leq \infty$ where $\tau_n$ is the restriction of $\tau$ on $\mathcal{M}_n$.

We remark that if $\mathcal{N}$ is a von Neumann subalgebra of $\mathcal{M}$, then there is a normal conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$ if and only if the restriction of the trace of $\mathcal{M}$ to $\mathcal{N}$ remains semi-finite. For the case where $\mathcal{M}$ is finite, such conditional expectations always exist. Indeed, if $\mathcal{N}$ is a von Neumann subalgebra of a finite von Neumann algebra $\mathcal{M}$, then the embedding $\iota$: $L^1(\mathcal{N}, \tau) \to L^1(\mathcal{M}, \tau)$ is an isometry and the dual map $\mathcal{E} = \iota^*$: $\mathcal{M} \to \mathcal{N}$ yields a conditional expectation (see, for instance, [T, Theorem 3.4]).

The following definition isolates the main topic of this paper.

*Definition 1.1:* A non-commutative martingale with respect to the filtration $(\mathcal{M}_n)_{n=1}^\infty$ is a sequence $x = (x_n)_{n=1}^\infty$ in $L^1(\mathcal{M}, \tau)$ such that

$$\mathcal{E}_n(x_{n+1}) = x_n \quad \text{for all } n \geq 1.$$

Similarly, if for all $n \geq 1$, $\mathcal{E}_n(x_{n+1}) \leq x_n$ (respectively, $\mathcal{E}_n(x_{n+1}) \geq x_n$), then $(x_n)_{n=1}^\infty$ is called a supermartingale (respectively, submartingale).

If additionally $x \in L^p(\mathcal{M}, \tau)$ for some $1 < p < \infty$, then $x$ is called a $L^p$-martingale. In this case, we set

$$\|x\|_p := \sup_{n \geq 1} \|x_n\|_p.$$

If $\|x\|_p < \infty$, then $x$ is called a bounded $L^p$-martingale. The difference sequence $dx = (dx_n)_{n=1}^\infty$ of a martingale (or just a general sequence) $x = (x_n)_{n=1}^\infty$ is defined by

$$dx_n = x_n - x_{n-1}$$

with the usual convention that $x_0 = 0$.

Recall that a subset $K$ of $L^1(\mathcal{M}, \tau)$ is said to be uniformly integrable if it is bounded and for every sequence of projections $(p_n)_{n=1}^\infty$ with $p_n \downarrow_n 0$, we have $\lim_{n\to\infty} \sup\{\|p_n h p_n\|_1; h \in K\} = 0$ ([R2]). It is clear that a martingale $x = (x_n)_{n=1}^\infty$ in $L^1(\mathcal{M}, \tau)$ is uniformly integrable if and only if there exists $x_\infty \in L^1(\mathcal{M}, \tau)$ such that $x_n = \mathcal{E}_n(x_\infty)$ for all $n \geq 1$. In this case, the sequence

$(x_n)_{n=1}^\infty$ converges to $x_\infty$ in $L^1(\mathcal{M},\tau)$. In particular, if $1<p<\infty$, then every bounded $L^p$-martingale is of the form $(\mathcal{E}_n(x_\infty))_{n=1}^\infty$ for some $x_\infty \in L^p(\mathcal{M},\tau)$.

The following decomposition of bounded $L^1$-martingale is the non-commutative extension of the classical Krickeberg's decomposition of martingales into linear combinations of positive martingales. It will be used in the sequel.

THEOREM 1.2 ([C]): *Let $x=(x_n)_{n=1}^\infty$ be a bounded $L^1$-martingale. Then $(x_n)_{n=1}^\infty$ admits the following decomposition:*

$$x_n = (x_n^{(1)} - x_n^{(2)}) + i(x_n^{(3)} - x_n^{(4)})$$

*for all $n\geq 1$ where, for each $j\in\{1,2,3,4\}$, the sequence $(x_n^{(j)})_{n=1}^\infty$ is a positive bounded $L^1$-martingale. Moreover, if $x_n = x_n^*$, for all $n\geq 1$, then $\|x\|_1 = \tau(x_1^{(1)}) + \tau(x_1^{(2)})$.*

The proposition below can be viewed as a substitute for the classical weak-type $(1,1)$ boundedness of maximal functions. It plays a crucial role in the proof of our main result. A short proof of the form stated below can be found in [R1].

PROPOSITION 1.3 ([C]): *If $x=(x_n)_{n=1}^\infty$ is a positive bounded $L^1$-martingale and $\lambda>0$, then there exists a sequence of decreasing projections $(q_n^{(\lambda)})_{n=1}^\infty$ in $\mathcal{M}$ with:*

- (i) *for every $n\geq 1$, $q_n^{(\lambda)}\in\mathcal{M}_n$;*
- (ii) *$q_n^{(\lambda)}$ commutes with $q_{n-1}^{(\lambda)}x_n q_{n-1}^{(\lambda)}$;*
- (iii) *$q_n^{(\lambda)}x_n q_n^{(\lambda)} \leq \lambda q_n^{(\lambda)}$;*
- (iv) *$(q_n^{(\lambda)})_{n=1}^\infty$ is a decreasing sequence and if we set $q^{(\lambda)} = \bigwedge_{n=1}^\infty q_n^{(\lambda)}$ then $\tau(\mathbf{1}-q^{(\lambda)}) \leq \|x\|_1/\lambda$.*

A "disjoint"-version of Proposition 1.3 as described below will be used in the sequel.

PROPOSITION 1.4: *If $x=(x_n)_{n=1}^\infty$ is a positive bounded $L^1$-martingale and $q_n^{(\lambda)}$ is the projection associated to $\lambda>0$ and $n\geq 1$ as in Proposition 1.3, then there exists a sequence of disjoint projections $(p_{i,n})_{i=0}^\infty$ in $\mathcal{M}_n$ with the following properties:*

- (i) *$\sum_{i=0}^\infty p_{i,n} = \mathbf{1}$ for the strong operator topology;*
- (ii) *for every $n_0\geq 1$, $\sum_{i=0}^{n_0} p_{i,n} \leq q_n^{(2^{n_0})}$.*

*Proof:* We will use Proposition 1.3 inductively. Set $p_{0,n} = \bigwedge_{k=0}^\infty q_n^{(2^k)}$ and, for $i\geq 1$,

$$p_{i,n} := \bigwedge_{k=i}^\infty q_n^{(2^k)} - \bigwedge_{k=i-1}^\infty q_n^{(2^k)}. \tag{1.1}$$

Clearly $(p_{i,n})_{i=0}^{\infty}$ is a sequence of disjoint projections in $\mathcal{M}_n$. Moreover, for every $m \geq 1$, $\sum_{i=0}^{m} p_{i,n} = \bigwedge_{k=m}^{\infty} q_n^{(2^k)}$ so

$$\begin{aligned}\tau(\mathbf{1} - \sum_{i=0}^{m} p_{i,n}) &= \tau(\mathbf{1} - \bigwedge_{k=m}^{\infty} q_n^{(2^k)}) \\ &= \tau(\bigvee_{k=m}^{\infty} (\mathbf{1} - q_n^{(2^k)})) \\ &\leq \sum_{k=m}^{\infty} \tau(\mathbf{1} - q_n^{(2^k)}) \leq \sum_{k=m}^{\infty} 2^{-k}\|x\|_1\end{aligned}$$

which proves that $\tau(\mathbf{1} - \sum_{i=0}^{\infty} p_{i,n}) = \lim_{m\to\infty} \tau(\mathbf{1} - \sum_{i=0}^{m} p_{i,n}) = 0$. Notice that $(\mathbf{1} - \sum_{i=0}^{m} p_{i,n})_{m=0}^{\infty}$ is a decreasing sequence of projections in the finite von Neumann subalgebra $(\mathbf{1} - p_{0,n})\mathcal{M}(\mathbf{1} - p_{0,n})$ and, since $\tau$ restricted to $(\mathbf{1} - p_{0,n})\mathcal{M}(\mathbf{1} - p_{0,n})$ is a faithful normal functional, we can conclude that $\mathbf{1} - \sum_{i=0}^{m} p_{i,n} \downarrow_m 0$ and hence $\sum_{i=0}^{\infty} p_{i,n} = \mathbf{1}$. To conclude the proof, note that as $\sum_{i=0}^{n_0} p_{i,n} = \bigwedge_{k=n_0}^{\infty} q_n^{(2^k)}$, it is clear that it is a subprojection of $q_n^{(2^{n_0})}$. The proof of the proposition is complete. ∎

*Remark 1.5:* In the statement of Proposition 1.4, one can also use the projections $q^{(2^k)}$ for $k \geq 1$. That is, if we set $p_0 = \bigwedge_{k=0}^{\infty} q^{(2^k)}$ and for $i \geq 1$, $p_i := \bigwedge_{k=i}^{\infty} q^{(2^k)} - \bigwedge_{k=i-1}^{\infty} q^{(2^k)}$, then $(p_i)_{i=1}^{\infty}$ is a sequence of disjoint projections in $\mathcal{M}$ with:

(i) $\sum_{i=0}^{\infty} p_i = \mathbf{1}$ for the strong operator topology;

(ii) for every $n_0 \geq 1$, $\sum_{i=0}^{n_0} p_i \leq q^{(2^{n_0})}$.

We end this section by recalling the general definition of non-commutative spaces associated with rearrangement invariant spaces. For this purpose, we consider the general construction of non-commutative spaces as sets of densely defined operators on a Hilbert space. This allows an easy introduction of non-commutative weak-$L^p$-spaces that are central for the topic of this paper.

Throughout, $H$ will denote a Hilbert space and $\mathcal{M} \subseteq B(H)$. A closed densely defined operator $a$ on $H$ is said to be **affiliated with** $\mathcal{M}$ if $u^*au = a$ for all unitary $u$ in the commutant $\mathcal{M}'$ of $\mathcal{M}$. If $a$ is a densely defined self-adjoint operator on $H$, and if $a = \int_{-\infty}^{\infty} s de_s^a$ is its spectral decomposition, then for any Borel subset $B \subseteq \mathbb{R}$, we denote by $\chi_B(a)$ the corresponding spectral projection $\int_{-\infty}^{\infty} \chi_B(s) de_s^a$. A closed densely defined operator $a$ on $H$ affiliated with $\mathcal{M}$ is said to be $\tau$**-measurable** if there exists a number $s \geq 0$ such that $\tau(\chi_{(s,\infty)}(|a|)) < \infty$.

The set of all $\tau$-measurable operators will be denoted by $\overline{\mathcal{M}}$. The set $\overline{\mathcal{M}}$ is a $*$-algebra with respect to the strong sum, the strong product, and the adjoint operation [N]. For $x \in \overline{\mathcal{M}}$, the generalized singular value function $\mu(x)$ of $x$ is defined by

$$\mu_t(x) = \inf\{s \geq 0: \tau(\chi_{(s,\infty)}(|x|)) \leq t\}, \quad \text{for } t \geq 0.$$

The function $t \to \mu_t(x)$ from the interval $[0, \tau(\mathbf{1}))$ to $[0, \infty)$ is right continuous, non-increasing and is the inverse of the distribution function $\lambda(x)$, where $\lambda_s(x) = \tau(\chi_{(s,\infty)}(|x|))$, for $s \geq 0$. For an in-depth study of $\mu(.)$ and $\lambda(.)$, we refer the reader to [FK]. For the definition below, we refer the reader to [BS] and [LT] for the theory of rearrangement invariant function spaces.

*Definition 1.6:* Let $E$ be a rearrangement invariant (quasi-) Banach function space on the interval $[0, \tau(\mathbf{1}))$. We define the symmetric space $E(\mathcal{M}, \tau)$ of measurable operators by setting

$$E(\mathcal{M}, \tau) = \{x \in \overline{\mathcal{M}}: \mu(x) \in E\} \quad \text{and}$$
$$\|x\|_{E(\mathcal{M},\tau)} = \|\mu(x)\|_E, \quad \text{for } x \in E(\mathcal{M}, \tau).$$

It is well known that $E(\mathcal{M}, \tau)$ is a Banach space (respectively, quasi-Banach space) if $E$ is a Banach space (respectively, quasi-Banach space). The space $E(\mathcal{M}, \tau)$ is often referred to as the non-commutative analogue of the function space $E$ and if $E = L^p[0, \tau(\mathbf{1}))$, for $0 < p \leq \infty$, then $E(\mathcal{M}, \tau)$ coincides with the usual non-commutative $L^p$-space associated with $(\mathcal{M}, \tau)$. We refer to [CS], [DDP1], [DDP2] and [X] for more detailed discussions about these spaces. Of special interest in this paper is the non-commutative weak $L^p$-spaces. For $0 < p < \infty$, the non-commutative weak $L^p$-space, denoted by $L^{p,\infty}(\mathcal{M}, \tau)$, is defined as the linear subspace of all $x \in \overline{\mathcal{M}}$ for which the quasi-norm

$$\|x\|_{p,\infty} := \sup_{t>0} t^{1/p}\mu_t(x) = \sup_{\lambda>0} \lambda\tau(\chi_{(\lambda,\infty)}(|x|))^{1/p}$$

is finite. Equipped with the quasi-norm $\|\cdot\|_{p,\infty}$, $L^{p,\infty}(\mathcal{M}, \tau)$ is a quasi-Banach space. It is easy to verify that if $0 < p < \infty$, then $\|x\|_{p,\infty} \leq \|x\|_p$ for all $x \in L^p(\mathcal{M}, \tau)$, and if $\tau$ is a normalized finite trace, then for $0 < r < p < \infty$, $\|y\|_r \leq \|y\|_{p,\infty}$ for all $y \in L^{p,\infty}(\mathcal{M}, \tau)$.

## 2. The main result

Following the lead of Pisier and Xu [PX], we will consider the following square functions: for a sequence $x = (x_n)_{n=1}^{\infty}$ (not necessarily a martingale), we denote by $dx$ the difference sequence as in Section 1. For $n \geq 1$, set

$$S_{C,n}(x) = \left(\sum_{k=1}^{n} |dx_k|^2\right)^{1/2} \quad \text{and} \quad S_{R,n}(x) = \left(\sum_{k=1}^{n} |dx_k^*|^2\right)^{1/2}.$$

Let $E[0,\tau(\mathbf{1}))$ be a rearrangement invariant (quasi-) Banach function space on $[0,\tau(\mathbf{1}))$. For any finite sequence $a=(a_n)_{n\geq 1}$ in $E(\mathcal{M},\tau)$, set

$$\|a\|_{E(\mathcal{M};l^2_C)} = \left\| \left( \sum_{n\geq 1} |a_n|^2 \right)^{1/2} \right\|_{E(\mathcal{M},\tau)}$$

and

$$\|a\|_{E(\mathcal{M};l^2_R)} = \left\| \left( \sum_{n\geq 1} |a_n^*|^2 \right)^{1/2} \right\|_{E(\mathcal{M},\tau)}.$$

The difference sequence $dx$ belongs to $E(\mathcal{M};l^2_C)$ (respectively, $E(\mathcal{M};l^2_R)$) if and only if the sequence $(S_{C,n}(x))_{n=1}^\infty$ (respectively, $(S_{R,n}(x))_{n=1}^\infty$) is a bounded sequence in $E(\mathcal{M},\tau)$. In this case, the limits $S_C(x) = (\sum_{k=1}^\infty |dx_k|^2)^{1/2}$ and $S_R(x) = (\sum_{k=1}^\infty |dx_k^*|^2)^{1/2}$ are elements of $E(\mathcal{M},\tau)$.

We will retain all notations introduced in the preliminaries. In particular, all adapted sequences are understood to be with respect to a fixed filtration of von Neumann subalgebras. The principal result of this paper is Theorem 2.1 below which generalizes a classical result of Burkholder stated in Theorem 0.1.

THEOREM 2.1: *There is an absolute constant $K$ such that if $x=(x_n)_{n=1}^\infty$ is a $L^1$-bounded martingale, then there exist two sequences $y=(y_n)_{n=1}^\infty$ and $z=(z_n)_{n=1}^\infty$ such that:*

($\alpha$) *for every $n\geq 1$, $x_n = y_n + z_n$;*

($\beta$) *for every $n \geq 1$, $y_n$ and $z_n$ belong to $L^{1,\infty}(\mathcal{M}_n,\tau_n)$;*

($\gamma$) $\|dy\|_{L^{1,\infty}(\mathcal{M};l^2_C)} + \|dz\|_{L^{1,\infty}(\mathcal{M};l^2_R)} \leq K\|x\|_1$.

To emphasize the role played by the triangular truncations, we will treat first the case where the von Neumann algebra $\mathcal{M}$ is finite with the trace $\tau$ being normalized. Then we will sketch the necessary adjustments needed to recover the general semi-finite case. Of course one could directly present the proof for the semi-finite case, but the added technicalities would obscure the general philosophy behind the main construction.

Assume that $\mathcal{M}$ is a finite von Neumann algebra and $\tau$ is a finite normalized trace. The proof will be divided into several cases.

CASE A: *The martingale $x=(x_n)_{n=1}^\infty$ is a positive martingale and $\|x\|_1 = 1$.* This is the most important case as we will see below that the general case can be deduced from Case A via Theorem 1.2. We will use several steps.

STEP 1: *Construction of the sequences $(y_n)_{n=1}^\infty$ and $(z_n)_{n=1}^\infty$.*

For each $n\geq 1$, let $(p_{i,n})_{i=0}^\infty$ be the sequence of disjoint projections in $\mathcal{M}_n$ obtained from Proposition 1.4. Since $\sum_{i=0}^\infty p_{i,n} = \mathbf{1}$, we have that for every

$a \in L^1(\mathcal{M},\tau)$, we can write: $a = \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} p_{i,n}ap_{j,n}$. Define the sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ as follows: for $n \geq 1$,

$$\left\{\begin{array}{l} dy_n := \sum_{j=0}^{\infty}\sum_{i\leq j} p_{i,n}dx_np_{j,n} \quad \text{and} \\ dz_n := \sum_{j=0}^{\infty}\sum_{i>j} p_{i,n}dx_np_{j,n}. \end{array}\right. \tag{2.1}$$

Clearly, $dx_n = dy_n + dz_n$ for every $n \geq 1$ and therefore $x_n = y_n + z_n$. Moreover, as $dx_n \in L^1(\mathcal{M}_n,\tau_n)$ and $(p_{i,n})_{i=0}^{\infty}$ are disjoint projections in $\mathcal{M}_n$, $dy_n$ and $dz_n$ are triangular truncations of $dx_n$, we get that $dy_n$ and $dz_n$ belong to $L^{1,\infty}(\mathcal{M}_n,\tau_n)$ (see [DDPS, Theorem 1.4] or [R3, Theorem 4.8]) and so do $y_n$ and $z_n$.

We will only have to show that there is an absolute constant $C$ such that

$$\|dy\|_{L^{1,\infty}(\mathcal{M};l_C^2)} \leq C\|x\|_1. \tag{2.2}$$

Indeed, if such inequality is valid, it follows from the construction of $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ that $\|dz\|_{L^{1,\infty}(\mathcal{M};l_R^2)}$ should satisfy the same inequality. In fact, $\sum_{n=1}^{\infty}|dz_n^*|^2$ and $\sum_{n=1}^{\infty}|dy_n|^2$ are essentially of the same form as demonstrated in the next lemma.

LEMMA 2.2: *For every $n \geq 1$, we have*

$$|dy_n|^2 = \sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i\leq\min(l,j)} p_{l,n}dx_np_{i,n}dx_np_{j,n} \quad \text{and}$$
$$|dz_n^*|^2 = \sum_{l=1}^{\infty}\sum_{j=1}^{\infty}\sum_{i<\min(l,j)} p_{l,n}dx_np_{i,n}dx_np_{j,n}$$

*where the sums are taken in the measure topology.*

*Proof:* For $n \geq 1$ and $N \geq 1$, set $dy_n^{(N)} = \sum_{j=0}^{N}\sum_{i\leq j} p_{i,n}dx_np_{j,n}$. Clearly, $dy_n^{(N)} \in L^1(\mathcal{M},\tau)$ and $dy_n^{(N)}$ converges to $dy_n$ in $L^{1,\infty}(\mathcal{M},\tau)$ if $N \to \infty$. Consequently, $|dy_n^{(N)}|^2$ converges to $|dy_n|^2$ in $L^{1/2,\infty}(\mathcal{M},\tau)$ if $N \to \infty$. We have, by the definition of $dy_n^{(N)}$,

$$\begin{aligned} |dy_n^{(N)}|^2 &= \left(\sum_{l=0}^{N}\sum_{m\leq l} p_{l,n}dx_np_{m,n}\right)\left(\sum_{j=0}^{N}\sum_{i\leq j} p_{i,n}dx_np_{j,n}\right) \\ &= \sum_{l=0}^{N}\sum_{j=0}^{N} p_{l,n}dx_n\left(\sum_{m\leq l} p_{m,n}\right)\left(\sum_{i\leq j} p_{i,n}\right)dx_np_{j,n}. \end{aligned}$$

As the $p_{i,n}$'s are disjoint, $(\sum_{m\leq l} p_{m,n})(\sum_{i\leq j} p_{i,n}) = \sum_{i\leq \min(l,j)} p_{i,n}$ and therefore,

$$|dy_n^{(N)}|^2 = \sum_{l=0}^{N}\sum_{j=0}^{N}\sum_{i\leq \min(l,j)} p_{l,n}dx_n p_{i,n} dx_n p_{j,n}.$$

Taking the limit as $N \to \infty$, we obtain the expression of $|dy_n|^2$ as stated. Similarly, since $dz_n^* = \sum_{j=0}^{\infty}\sum_{i>j} p_{j,n}dx_n p_{i,n} = \sum_{i=1}^{\infty}\sum_{j<i} p_{j,n}dx_n p_{i,n}$, the same computation gives that $|dz_n^*|^2$ is as stated in the lemma. ∎

We are ready for the proof. According to the definition of the quasi-norm $\|\cdot\|_{1,\infty}$, we will have to show the existence of a numerical constant $C$ such that for every $\lambda > 0$,

$$\tau(\chi_{(\lambda,\infty)}(S_C(y))) \leq C\lambda^{-1}. \tag{2.3}$$

Some of the techniques used below were already employed in [R1]. However, because of the complexity of the proof, we choose to include all details. All steps taken below can be read independently from [R1].

♦ First, we consider the particular case: $\lambda = 2^{n_0}$ *for some* $n_0 \geq 0$.

Step 2: *Reduction to bounded difference sequence.* Until the end of the proof, we will simply write $(q_n)_{n=1}^{\infty}$ (respectively, $q$) for the projections $(q_n^{(2^{n_0})})_{n=1}^{\infty}$ (respectively, $q^{(2^{n_0})}$).

Lemma 2.3: *Let* $w_{n_0} = \sum_{i=0}^{n_0} p_i = \bigwedge_{k=n_0}^{\infty} q^{(2^k)}$. *For every* $\alpha \in (0,1)$ *and every* $\beta \in (0,1)$,

$$\tau(\chi_{(2^{n_0},\infty)}(S_C(y))) \leq \alpha^{-1}\beta^{-1}4^{-n_0}\tau(w_{n_0}S_C(y)^2 w_{n_0}) + 4(1-\alpha)^{-1}2^{-n_0}.$$

*Proof:* Set $S = S_C(y)^2 = \sum_{n=1}^{\infty}|dy_n|^2$. Split the operator $S$ into three parts:

$$S = w_{n_0}Sw_{n_0} + (\mathbf{1} - w_{n_0})Sw_{n_0} + S(\mathbf{1} - w_{n_0}).$$

Fix $\alpha \in (0,1)$ and $\beta \in (0,1)$. Using properties of generalized singular value functions $\mu(\cdot)$ from [FK], we have

$$\tau(\chi_{(2^{n_0},\infty)}(S^{1/2})) = \int_0^1 \chi_{(2^{n_0},\infty)}(\mu_t(S^{1/2}))\, dt.$$

This follows from [FK, Corollary 2.8] by approximating the characteristic function $\chi_{(2^{n_0},\infty)}(\cdot)$ from below by sequences of continuous functions $f$ on $[0,\infty)$ satisfying

$f(0)=0$. We can deduce the following estimate:

$$\begin{aligned}\tau(\chi_{(2^{n_0},\infty)}(S^{1/2})) &= \int_0^1 \chi_{(2^{n_0},\infty)}(\mu_t(S)^{1/2})\,dt\\ &= \int_0^1 \chi_{(4^{n_0},\infty)}(\mu_t(S))\,dt\\ &\le \int_0^1 \chi_{(4^{n_0},\infty)}\{\mu_{\alpha t}(w_{n_0}Sw_{n_0})+\mu_{(1-\alpha)t/2}((\mathbf{1}-w_{n_0})Sw_{n_0})\\ &\quad+\mu_{(1-\alpha)t/2}(S(\mathbf{1}-w_{n_0}))\}\,dt.\end{aligned}$$

As $\mu_{(1-\alpha)t/2}(w_{n_0}S(\mathbf{1}-w_{n_0}))\le\mu_{(1-\alpha)t/2}(|S(\mathbf{1}-w_{n_0})|)$, we get

$$\begin{aligned}\tau(\chi_{(2^{n_0},\infty)}(S^{1/2})) &\le \int_0^1 \chi_{(4^{n_0},\infty)}\{\mu_{\alpha t}(w_{n_0}Sw_{n_0})\\ &\quad+2\mu_{(1-\alpha)t/2}(|S(\mathbf{1}-w_{n_0})|)\}\,dt\\ &\le \int_0^1 \chi_{(\beta 4^{n_0},\infty)}\{\mu_{\alpha t}(w_{n_0}Sw_{n_0})\}\,dt\\ &\quad+\int_0^1 \chi_{((1-\beta)4^{n_0},\infty)}\{\mu_{(1-\alpha)t/2}(2|S(\mathbf{1}-w_{n_0})|)\}\,dt\\ &= \int_0^1 \mu_{\alpha t}(\chi_{(\beta 4^{n_0},\infty)}(w_{n_0}Sw_{n_0}))\,dt\\ &\quad+\int_0^1 \mu_{(1-\alpha)t/2}\{\chi_{((1-\beta)4^{n_0},\infty)}(2|S(\mathbf{1}-w_{n_0})|)\}\,dt.\end{aligned}$$

Since $\chi_{((1-\beta)4^{n_0},\infty)}(2|S(\mathbf{1}-w_{n_0})|)$ is a subprojection of $\mathbf{1}-w_{n_0}$, it follows that

$$\tau(\chi_{(2^{n_0},\infty)}(S^{1/2}))\le\int_0^1\mu_{\alpha t}(\chi_{(\beta 4^{n_0},\infty)}(w_{n_0}Sw_{n_0}))\,dt+\int_0^1\mu_{(1-\alpha)t/2}(\mathbf{1}-w_{n_0})\,dt$$

and, by change of variables,

$$\begin{aligned}\tau(\chi_{(2^{n_0},\infty)}(S^{1/2})) \le&\alpha^{-1}\int_0^1\mu_t(\chi_{(\beta 4^{n_0},\infty)}(w_{n_0}Sw_{n_0}))\,dt\\ &+2(1-\alpha)^{-1}\int_0^1\mu_t(\mathbf{1}-w_{n_0})\,dt,\end{aligned}$$

and therefore

$$\tau(\chi_{(2^{n_0},\infty)}(S^{1/2}))\le\alpha^{-1}\int_0^1\mu_t(\chi_{(\beta 4^{n_0},\infty)}(w_{n_0}Sw_{n_0}))\,dt+2(1-\alpha)^{-1}\tau(\mathbf{1}-w_{n_0}).$$

Recall that $w_{n_0}=\sum_{i=0}^{n_0}p_i=\bigwedge_{k=n_0}^{\infty}q^{(2^k)}$ so $\mathbf{1}-w_{n_0}=\bigvee_{k=n_0}^{\infty}(\mathbf{1}-q^{(2^k)})$. By

Proposition 1.3, we can deduce that

$$\tau(\mathbf{1} - w_{n_0}) \leq \sum_{k=n_0}^{\infty} \tau(\mathbf{1} - q^{(2^k)})$$
$$\leq \sum_{k=n_0}^{\infty} 2^{-k} = 2.2^{-n_0}.$$

Combining with the previous estimate, we have

$$\tau(\chi_{(2^{n_0},\infty)}(S^{1/2})) \leq \alpha^{-1} \int_0^1 \mu_t(\chi_{(\beta 4^{n_0},\infty)}(w_{n_0} S w_{n_0}))\, dt + 4(1-\alpha)^{-1} 2^{-n_0}.$$

We conclude the inequality,

$$\tau(\chi_{(2^{n_0},\infty)}(S^{1/2})) \leq \alpha^{-1}\beta^{-1}4^{-n_0}\tau(w_{n_0} S w_{n_0}) + 4(1-\alpha)^{-1}2^{-n_0}.$$

Thus the lemma is proved. ∎

STEP 3: *Difference sequence of a supermartingale in $L^2(\mathcal{M}, \tau)$.*

LEMMA 2.4: *The sequence $(q_n x_n q_n)_{n=1}^{\infty}$ is a supermartingale in $L^2(\mathcal{M}, \tau)$ and*

$$\tau(w_{n_0} S_C(y)^2 w_{n_0}) \leq \sum_{n=1}^{\infty} \|q_n x_n q_n - q_{n-1} x_{n-1} q_{n-1}\|_2^2.$$

*Proof:* We first note that since both sequences $(q_n)_{n=1}^{\infty}$ and $(x_n)_{n=1}^{\infty}$ are adapted, it is clear that $(q_n x_n q_n)_{n=1}^{\infty}$ is adapted. To prove that it is a supermartingale, we need to verify that for every $n \geq 2$, $\mathcal{E}_{n-1}(q_n x_n q_n) \leq q_{n-1} x_{n-1} q_{n-1}$. This follows from the construction of the sequence $(q_n)_{n=1}^{\infty}$ in Proposition 1.3. In fact, since $q_n$ commutes with $q_{n-1} x_n q_{n-1}$ and $q_n \leq q_{n-1}$, $q_n x_n q_n \leq q_{n-1} x_n q_{n-1}$. As $\mathcal{E}_{n-1}$ is a positive contraction,

$$\mathcal{E}_{n-1}(q_n x_n q_n) \leq \mathcal{E}_{n-1}(q_{n-1} x_n q_{n-1})$$
$$= q_{n-1}\mathcal{E}_{n-1}(x_n) q_{n-1}$$
$$= q_{n-1} x_{n-1} q_{n-1}.$$

This proves that $(q_n x_n q_n)_{n=1}^{\infty}$ is a supermartingale. To prove the inequality, fix $N \geq 1$. From the form of $|dy_n|^2$ stated in Lemma 2.2, we can write

$$w_{n_0} S_{C,N}^2(y) w_{n_0} = \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i \leq \min(l,j)} w_{n_0} p_{l,n} dx_n p_{i,n} dx_n p_{j,n} w_{n_0}.$$

We claim that the sums taken in the expression of $w_{n_0}S^2_{C,N}(y)w_{n_0}$ above are finite sums. For this, we remark that if $l > n_0$, then $w_{n_0}p_{l,n} = p_{l,n}w_{n_0} = 0$. In fact, as $p_{l,n} = \bigwedge_{k=l}^{\infty} q_n^{(2^k)} - \bigwedge_{k=l-1}^{\infty} q_n^{(2^k)}$ and $q^{(2^k)} \leq q_n^{(2^k)}$ for all $k \geq 1$, it is clear that $w_{n_0} = \bigwedge_{k=n_0}^{\infty} q^{(2^k)}$ is a subprojection of $\bigwedge_{k=l-1}^{\infty} q_n^{(2^k)}$ when $l > n_0$ and therefore $w_{n_0} \perp p_{l,n}$. With this observation, we can write

$$\begin{aligned} w_{n_0}S^2_{C,N}(y)w_{n_0} &= \sum_{n=1}^{N}\sum_{l=0}^{n_0}\sum_{j=0}^{n_0}\sum_{i\leq min(l,j)} w_{n_0}p_{l,n}dx_np_{i,n}dx_np_{j,n}w_{n_0} \\ &= \sum_{n=1}^{N} w_{n_0}\Big(\sum_{l=0}^{n_0}\sum_{j=0}^{n_0}\sum_{i\leq min(l,j)} p_{l,n}dx_np_{i,n}dx_np_{j,n}\Big)w_{n_0}. \end{aligned}$$

Taking the trace, we get

$$\begin{aligned} \tau(w_{n_0}S^2_{C,N}(y)w_{n_0}) &\leq \sum_{n=1}^{N}\sum_{l=0}^{n_0}\sum_{j=0}^{n_0}\sum_{i\leq min(l,j)} \tau(p_{l,n}dx_np_{i,n}dx_np_{j,n}) \\ &= \sum_{n=1}^{N}\sum_{l=0}^{n_0}\tau\Big(p_{l,n}dx_n(\sum_{i\leq l}p_{i,n})dx_np_{l,n}\Big) \\ &\leq \sum_{n=1}^{N}\sum_{l=0}^{n_0}\tau\Big(p_{l,n}dx_n\Big(\sum_{i=0}^{n_0}p_{i,n}\Big)dx_np_{l,n}\Big). \end{aligned}$$

Since $\sum_{i=0}^{n_0} p_{i,n} = \bigwedge_{k=n_0}^{\infty} q_n^{(2^k)} \leq q_n$, we have

$$\begin{aligned} \tau(w_{n_0}S^2_{C,N}(y)w_{n_0}) &\leq \sum_{n=1}^{N}\sum_{l=0}^{n_0}\tau(p_{l,n}dx_nq_ndx_np_{l,n}) \\ &= \sum_{n=1}^{N}\tau\Big(\Big(\sum_{l=0}^{n_0}p_{l,n}\Big)dx_nq_ndx_n\Big) \\ &= \sum_{n=1}^{N}\tau\Big(q_ndx_n\Big(\sum_{l=0}^{n_0}p_{l,n}\Big)dx_nq_n\Big) \\ &\leq \sum_{n=1}^{N}\tau(q_ndx_nq_ndx_nq_n). \end{aligned}$$

We can also get this by noting that from the computation of $w_{n_0}S^2_{C,N}(y)w_{n_0}$ above we have $w_{n_0}S^2_{C,N}(y)w_{n_0} = w_{n_0}(\sum_{n=1}^{N}|\sum_{j=0}^{n_0}\sum_{i\leq j}p_{i,n}dx_np_{j,n}|^2)w_{n_0}$, and therefore

$$\tau(w_{n_0}S^2_{C,N}(y)w_{n_0}) \leq \sum_{n=1}^{N}\Big\|\sum_{j=0}^{n_0}\sum_{i\leq j}p_{i,n}dx_np_{j,n}\Big\|_2^2,$$

and since triangular truncations are contractive in $L^2(\mathcal{M},\tau)$, we have

$$\tau(w_{n_0}S^2_{C,N}(y)w_{n_0}) \leq \sum_{n=1}^{N}\Big\|\sum_{j=0}^{n_0}\sum_{i=0}^{n_0} p_{i,n}dx_n p_{j,n}\Big\|_2^2$$
$$\leq \sum_{n=1}^{N}\|q_n dx_n q_n\|_2^2.$$

To conclude the proof, we will verify that for every $n \geq 1$,

$$\tau(q_n dx_n q_n dx_n q_n) \leq \|q_n x_n q_n - q_{n-1}x_{n-1}q_{n-1}\|_2^2.$$

This follows directly from the fact that $q_n \leq q_{n-1}$. In fact,

$$\begin{aligned}\tau(q_n dx_n q_n dx_n q_n) &= \tau(q_n(x_n - x_{n-1})q_n(x_n - x_{n-1})q_n)\\ &= \tau(|q_n(q_n x_n q_n - q_{n-1}x_{n-1}q_{n-1})q_n|^2)\\ &\leq \tau(q_n|q_n x_n q_n - q_{n-1}x_{n-1}q_{n-1}|^2 q_n)\\ &\leq \|q_n x_n q_n - q_{n-1}x_{n-1}q_{n-1}\|_2^2.\end{aligned}$$

Taking the limit as $N$ tends to $\infty$, the proof is complete. ∎

Note that if we set $\varrho = (q_n x_n q_n)_{n=1}^{\infty}$ then the right-hand side of the inequality in Lemma 2.4 is equal to $\|S_C(\varrho)\|_2^2$. This suggests the need to transform the supermartingale into a martingale.

STEP 4: *Write the supermartingale as a sum of a positive martingale in $L^2(\mathcal{M},\tau)$ and a decreasing sequence of operators.*

This is achieved by setting

$$\xi_n := \begin{cases} q_1 x_1 q_1 & \text{for } n = 1,\\ q_n x_n q_n + \sum_{l=1}^{n-1} q_l x_l q_l - \mathcal{E}_l(q_{l+1}x_{l+1}q_{l+1}) & \text{for } n \geq 2,\end{cases} \tag{2.4}$$

and

$$\zeta_n := \begin{cases} 0 & \text{for } n = 1,\\ \sum_{l=1}^{n-1} \mathcal{E}_l(q_{l+1}x_{l+1}q_{l+1}) - q_l x_l q_l & \text{for } n \geq 2.\end{cases} \tag{2.5}$$

Clearly, $\xi = (\xi_n)_{n=1}^{\infty}$ is a positive martingale. Moreover, for every $n \geq 1$,

$$\xi_n + \zeta_n = q_n x_n q_n \tag{2.6}$$

and

$$\zeta_n \leq \zeta_{n-1} \leq \cdots \leq \zeta_1 = 0. \tag{2.7}$$

LEMMA 2.5: *The sequence $\xi = (\xi_n)_{n=1}^{\infty}$ is a bounded $L^2$-martingale with*

$$\|\xi\|_2^2 \leq 3.2^{n_0+1}.$$

*Proof:* Since $\xi$ is a martingale, for every $N \geq 1$, we have

$$\|\xi_N\|_2^2 = \left\| \left( \sum_{n=1}^{N} |d\xi_n|^2 \right)^{1/2} \right\|_2^2 = \sum_{n=1}^{N} \|d\xi_n\|_2^2.$$

The main idea is to estimate $\|d\xi_n\|_2^2$ for all $n \geq 1$. Fix $n \geq 2$. We remark from the definition of $\xi$ in equation (2.4) that $\xi_n = \xi_{n-1} + q_n x_n q_n - \mathcal{E}_{n-1}(q_n x_n q_n)$ and therefore

$$\begin{aligned} d\xi_n &= q_n x_n x_n - \mathcal{E}_{n-1}(q_n x_n q_n) \\ &= (q_n x_n q_n - q_{n-1} x_{n-1} q_{n-1}) + (q_{n-1} x_{n-1} q_{n-1} - \mathcal{E}_{n-1}(q_n x_n q_n)). \end{aligned}$$

Since $\|\cdot\|_2^2$ is convex,

$$\begin{aligned} \|d\xi_n\|_2^2 \leq & 2(\|q_n x_n q_n - q_{n-1} x_{n-1} q_{n-1}\|_2^2 + \|q_{n-1} x_{n-1} q_{n-1} - \mathcal{E}_{n-1}(q_n x_n q_n)\|_2^2) \\ =& 2\tau((q_n x_n q_n - q_{n-1} x_{n-1} q_{n-1})^2) \\ &+ 2\tau((q_{n-1} x_{n-1} q_{n-1} - \mathcal{E}_{n-1}(q_n x_n q_n))^2) \\ =& I + II. \end{aligned}$$

We will estimate $I$ and $II$ separately. First for $I$, we use the identity $(a-b)^2 = a^2 - b^2 + b(b-a) + (b-a)b$ for self-adjoint operators. With $a = q_n x_n q_n$ and $b = q_{n-1} x_{n-1} q_{n-1}$, we have, after taking the trace,

$$\begin{aligned} I =& 2\tau((q_n x_n q_n)^2 - (q_{n-1} x_{n-1} q_{n-1})^2) \\ &+ 4\tau(q_{n-1} x_{n-1} q_{n-1}[q_{n-1} x_{n-1} q_{n-1} - q_n x_n q_n]) \\ =& 2\tau((q_n x_n q_n)^2 - (q_{n-1} x_{n-1} q_{n-1})^2) \\ &+ 4\tau(q_{n-1} x_{n-1} q_{n-1}[q_{n-1} x_{n-1} q_{n-1} - \mathcal{E}_{n-1}(q_n x_n q_n)]). \end{aligned}$$

By Proposition 1.3 (iii), $\|q_{n-1} x_{n-1} q_{n-1}\|_\infty \leq 2^{n_0}$. Moreover, as $(q_n x_n q_n)_{n=1}^{\infty}$ is a supermartingale, $q_{n-1} x_{n-1} q_{n-1} - \mathcal{E}_{n-1}(q_n x_n q_n) \geq 0$. Therefore, we get

$$\begin{aligned} I \leq & 2\tau((q_n x_n q_n)^2 - (q_{n-1} x_{n-1} q_{n-1})^2) \\ &+ 2^{n_0+2}\tau(q_{n-1} x_{n-1} q_{n-1} - \mathcal{E}_{n-1}(q_n x_n q_n)) \\ =& 2\tau((q_n x_n q_n)^2 - (q_{n-1} x_{n-1} q_{n-1})^2) + 2^{n_0+2}\tau(q_{n-1} x_{n-1} q_{n-1} - q_n x_n q_n). \end{aligned}$$

For $II$, again since $q_{n-1} x_{n-1} q_{n-1} \geq q_{n-1} x_{n-1} q_{n-1} - \mathcal{E}_{n-1}(q_n x_n q_n) \geq 0$, we have

$$\|q_{n-1} x_{n-1} q_{n-1} - \mathcal{E}_{n-1}(q_n x_n q_n)\|_\infty \leq \|q_{n-1} x_{n-1} q_{n-1}\|_\infty \leq 2^{n_0}.$$

Hence, we get

$$\begin{aligned} II &\leq 2^{n_0+1}\tau(q_{n-1}x_{n-1}q_{n-1} - \mathcal{E}_{n-1}(q_nx_nq_n)) \\ &= 2^{n_0+1}\tau(q_{n-1}x_{n-1}q_{n-1} - q_nx_nq_n). \end{aligned}$$

Combining the preceding estimates on $I$ and $II$, we conclude that for every $n \geq 2$,

$$\begin{aligned} \|d\xi_n\|_2^2 \leq &2(\|q_nx_nq_n\|_2^2 - \|q_{n-1}x_{n-1}q_{n-1}\|_2^2) \\ &+ 3.2^{n_0+1}\tau(q_{n-1}x_{n-1}q_{n-1} - q_nx_nq_n). \end{aligned}$$

Now, we take the summation over $1 \leq n \leq N$,

$$\begin{aligned} \|\xi_N\|_2^2 =&\|q_1x_1q_1\|_2^2 + \sum_{n=2}^{N} \|d\xi_n\|_2^2 \\ \leq&\|q_1x_1q_1\|_2^2 + 2\sum_{n=2}^{N}(\|q_nx_nq_n\|_2^2 - \|q_{n-1}x_{n-1}q_{n-1}\|_2^2) \\ &+ 3.2^{n_0+1}\sum_{n=2}^{N}\tau(q_{n-1}x_{n-1}q_{n-1} - q_nx_nq_n) \\ =&\|q_1x_1q_1\|_2^2 + 2(\|q_Nx_Nq_N\|_2^2 - \|q_1x_1q_1\|_2^2) \\ &+ 3.2^{n_0+1}\tau(q_1x_1q_1 - q_Nx_Nq_N) \\ =&2\|q_Nx_Nq_N\|_2^2 - \|q_1x_1q_1\|_2^2 + 3.2^{n_0+1}\tau((q_1 - q_N)x_N) \\ \leq&2^{n_0+1}\tau(q_Nx_N) - \|q_1x_1q_1\|_2^2 + 3.2^{n_0+1}\tau((q_1 - q_N)x_N) \\ =&3.2^{n_0+1}\tau(q_1x_N) - 2^{n_0+2}\tau(q_Nx_N) - \|q_1x_1q_1\|_2^2 \leq 3.2^{n_0+1}. \end{aligned}$$

Taking the limit as $N$ tends to $\infty$, the proof is complete. ∎

LEMMA 2.6: $\sum_{n=1}^{\infty} \|q_nx_nq_n - q_{n-1}x_{n-1}q_{n-1}\|_2^2 \leq 4\|\xi\|_2^2$.

*Proof:* Let $(r_n(\cdot))_{n=1}^{\infty}$ be the sequence of Rademacher functions on $[0,1]$ (see for instance [W, p. 12] for the definition and properties of $(r_n(\cdot))_{n=1}^{\infty}$). Let $N \geq 1$. From equation (2.6), we have

$$\begin{aligned} \sum_{n=1}^{N} \|q_nx_nq_n - q_{n-1}x_{n-1}q_{n-1}\|_2^2 &= \sum_{n=1}^{N} \|d\xi_n + d\zeta_n\|_2^2 \\ &= \int_0^1 \left\| \sum_{n=1}^{N} r_n(t)(d\xi_n + d\zeta_n)\right\|_2^2 dt. \end{aligned}$$

Since $\|\cdot\|_2^2$ is convex and martingale transforms are $L^2$-bounded,

$$\begin{aligned}\sum_{n=1}^{N}\|q_nx_n - q_{n-1}x_{n-1}q_{n-1}\|_2^2 \le & 2\int_0^1\left\|\sum_{n=1}^{N} r_n(t)d\xi_n\right\|_2^2 dt \\ & + 2\int_0^1\left\|\sum_{n=1}^{N} r_n(t)d\zeta_n\right\|_2^2 dt \\ = & 2\|\xi_N\|_2^2 + 2\int_0^1\left\|\sum_{n=1}^{N} r_n(t)d\zeta_n\right\|_2^2 dt \\ = & 2\|\xi_N\|_2^2 + III.\end{aligned}$$

To estimate $III$, recall from inequality (2.7) that $\zeta = (\zeta_n)_{n=1}^{\infty}$ is a decreasing sequence of operators with $\zeta_1 = 0$, so $d\zeta_n \le 0$ for all $n \ge 1$. For every $t \in [0,1]$,

$$\left|\sum_{n=1}^{N} r_n(t)d\zeta_n\right|^2 = \sum_{k=1}^{N}\sum_{n=1}^{N} r_n(t)r_k(t)(\zeta_k - \zeta_{k-1})(\zeta_n - \zeta_{n-1}).$$

So taking the trace on both sides,

$$\begin{aligned}\left\|\sum_{n=1}^{N} r_n(t)d\zeta_n\right\|_2^2 &= \sum_{k=1}^{N}\sum_{n=1}^{N}\tau(r_n(t)r_k(t)(\zeta_{k-1} - \zeta_k)(\zeta_{n-1} - \zeta_n)) \\ &= \sum_{k=1}^{N}\sum_{n=1}^{N}\tau((\zeta_{n-1} - \zeta_n)^{1/2} r_n(t)r_k(t)(\zeta_{k-1} - \zeta_k)(\zeta_{n-1} - \zeta_n)^{1/2}) \\ &\le \sum_{k=1}^{N}\sum_{n=1}^{N}\tau((\zeta_{n-1} - \zeta_n)^{1/2}(\zeta_{k-1} - \zeta_k)(\zeta_{n-1} - \zeta_n)^{1/2}) \\ &= \sum_{k=1}^{N}\sum_{n=1}^{N}\tau((\zeta_{k-1} - \zeta_k)(\zeta_{n-1} - \zeta_n)) \\ &= \tau\left(\left|\sum_{n=1}^{N} d\zeta_n\right|^2\right) \\ &= \|\zeta_N\|_2^2.\end{aligned}$$

Taking the integral, we have $III \le 2\|\zeta_N\|_2^2$ and therefore

$$\sum_{n=1}^{N}\|q_nx_nq_n - q_{n-1}x_{n-1}q_{n-1}\|_2^2 \le 2\|\xi_N\|_2^2 + 2\|\zeta_N\|_2^2.$$

To conclude the lemma, note that $\xi_N - q_Nx_Nq_N = -\zeta_N \ge 0$, so $\xi_N \ge -\zeta_N \ge 0$ which gives that $\|\xi_N\|_2 \ge \|\zeta_N\|_2$. Taking the limit as $N$ tends to $\infty$, the proof of the lemma is complete. ∎

To conclude the proof of inequality (2.3) for the case $\lambda = 2^{n_0}$, it is enough to put together Lemma 2.3, Lemma 2.4, Lemma 2.5 and Lemma 2.6 above. In fact,

$$\begin{aligned}\tau(\chi_{(2^{n_0},\infty)}(S_C(y))) \leq& \alpha^{-1}\beta^{-1}4^{-n_0}\tau(w_{n_0}S_C(y)^2 w_{n_0}) + 4(1-\alpha)^{-1}2^{-n_0}\\ \leq& \alpha^{-1}\beta^{-1}4^{-n_0}\sum_{n=1}^{\infty}\|q_n x_n q_n - q_{n-1}x_{n-1}q_{n-1}\|_2^2\\ &+ 4(1-\alpha)^{-1}2^{-n_0}\\ \leq& \alpha^{-1}\beta^{-1}4^{-n_0}(4\|\xi\|_2^2) + 4(1-\alpha)^{-1}2^{-n_0}\\ \leq& 24\alpha^{-1}\beta^{-1}2^{-n_0} + 4(1-\alpha)^{-1}2^{-n_0}.\end{aligned}$$

If we set $C_1 := \inf\{24\alpha^{-1}\beta^{-1} + 4(1-\alpha)^{-1}; \alpha \in (0,1), \beta \in (0,1)\}$ then

$$\tau(\chi_{(2^{n_0},\infty)}(S_C(y))) \leq C_1 2^{-n_0}.$$

Hence inequality (2.3) is verified for $\lambda = 2^{n_0}$.

♦ Assume now that $1 \leq \lambda < \infty$.
Fix $n_0 \geq 0$ such that $2^{n_0} \leq \lambda < 2^{n_0+1}$. We have

$$\chi_{(\lambda,\infty)}(S_C(y)) \leq \chi_{(2^{n_0},\infty)}(S_C(y)),$$

and therefore

$$\tau(\chi_{(\lambda,\infty)}(S_C(y))) \leq C_1 2^{-n_0} = 2C_1 2^{-(n_0+1)} \leq 2C_1\lambda^{-1}$$

which shows that inequality (2.3) is valid for $1 \leq \lambda < \infty$.

♦ For the case $0 < \lambda \leq 1$, we note that since $\tau(1) = 1$, $\tau(\chi_{(\lambda,\infty)}(S_C(y))) \leq 1$. In particular, $\tau(\chi_{(\lambda,\infty)}(S_C(y))) \leq \lambda^{-1}$. Hence inequality (2.3) is satisfied.

With all possible values of $\lambda$ now covered, we can now conclude

$$\|dy\|_{L^{1,\infty}(\mathcal{M};l_C^2)} \leq 2C_1.$$

From the similarity of $|dy_n|^2$ and $|dz_n^*|^2$ demonstrated in Lemma 2.3, we have

$$\|dy\|_{L^{1,\infty}(\mathcal{M};l_C^2)} + \|dz\|_{L^{1,\infty}(\mathcal{M};l_R^2)} \leq 4C_1.$$

This completes the proof of Case A with $C = 4C_1$. ∎

CASE B: *The martingale $x = (x_n)_{n=1}^{\infty}$ is a positive martingale.* For each $n \geq 1$, set $\widetilde{x}_n = x_n/\|x\|_1$. Then $\widetilde{x} = (\widetilde{x})_{n=1}^{\infty}$ is a positive martingale with $\|\widetilde{x}\|_1 = 1$. By Case A, there are sequences $\widetilde{y} = (\widetilde{y}_n)_{n=1}^{\infty}$ and $\widetilde{z} = (\widetilde{z}_n)_{n=1}^{\infty}$ with:

(i) for every $n \geq 1$, $\widetilde{x}_n = \widetilde{y}_n + \widetilde{z}_n$;

(ii) $\|d\widetilde{y}\|_{L^{1,\infty}(\mathcal{M};l^2_C)} + \|d\widetilde{z}\|_{L^{1,\infty}(\mathcal{M};l^2_R)} \leq C$.

Setting $y_n := \|x\|_1 \widetilde{y}_n$ and $z_n := \|x\|_1 \widetilde{z}_n$ for $n \geq 1$, we get that

$$\|dy\|_{L^{1,\infty}(\mathcal{M};l^2_C)} + \|dz\|_{L^{1,\infty}(\mathcal{M};l^2_R)} \leq C\|x\|_1.$$

This completes the proof of Case B. ∎

Case C: *The martingale $x = (x_n)_{n=1}^\infty$ is a $L^1$-bounded martingale with $x_n = x_n^*$ for all $n \geq 1$.* From Theorem 1.2, we can decompose the martingale $x$ into two positive $L^1$-bounded martingales $x^{(1)} = (x_n^{(1)})_{n=1}^\infty$ and $x^{(2)} = (x_n^{(2)})_{n=1}^\infty$ with $x_n = x_n^{(1)} - x_n^{(2)}$ for all $n \geq 1$ and $\|x\|_1 = \|x^{(1)}\|_1 + \|x^{(2)}\|_1$. For $j \in \{1,2\}$, let $(y^{(j)})_{n=1}^\infty$ and $(z_n^{(j)})_{n=1}^\infty$ be the decomposition of $x^{(j)} = (x_n^{(j)})_n$ as in Case B. For $n \geq 1$, we set

$$y_n := y_n^{(1)} - y_n^{(2)} \quad \text{and} \quad z_n := z_n^{(1)} - z_n^{(2)}.$$

We claim that $\|dy\|_{L^{1,\infty}(\mathcal{M};l^2_C)} + \|dz\|_{L^{1,\infty}(\mathcal{M};l^2_R)} \leq 4\sqrt{2}C\|x\|_1$ where $C$ is the constant from Case B.

For this, note that for every $n \geq 1$, $dy_n = dy_n^{(1)} - dy_n^{(2)}$ and therefore

$$\begin{aligned} |dy_n|^2 &= |dy_n^{(1)}|^2 - dy_n^{(1)}dy_n^{(2)} - dy_n^{(2)}dy_n^{(1)} + |dy_n^{(2)}|^2 \\ &\leq 2|dy_n^{(1)}|^2 + 2|dy_n^{(2)}|^2. \end{aligned}$$

Hence, $S_C(y)^2 \leq 2S_C(y^{(1)})^2 + 2S_C(y^{(2)})^2$. Using properties of generalized singular value functions [FK], we have, for every $t \in [0,1)$,

$$\begin{aligned} \mu_t(S_C(y)) &= \mu_t(S_C(y)^2)^{1/2} \\ &\leq \sqrt{2}\mu_t(S_C(y^{(1)})^2 + S_C(y^{(2)})^2)^{1/2} \\ &\leq \sqrt{2}\{\mu_{t/2}(S_C(y^{(1)})^2) + \mu_{t/2}(S_C(y^{(2)})^2\}^{1/2} \\ &= \sqrt{2}\{\mu_{t/2}(S_C(y^{(1)}))^2 + \mu_{t/2}(S_C(y^{(2)})^2\}^{1/2}. \end{aligned}$$

From Case B, we note from the definition of $\|\cdot\|_{1,\infty}$ that for $j \in \{1,2\}$ and $s \in [0,1)$, $\mu_s(S_C(y^{(j)})) \leq s^{-1}C\|x^{(j)}\|_1$. This implies that

$$\mu_t(S_C(y)) \leq \sqrt{2}\{4t^{-2}C^2(\|x^{(1)}\|_1^2 + \|x^{(2)}\|_1^2)\}^{1/2} \leq 2\sqrt{2}Ct^{-1}\|x\|_1.$$

Consequently, $\|dy\|_{L^{1,\infty}(\mathcal{M};l^2_C)} \leq 2\sqrt{2}C\|x\|_1$. A similar estimate can be performed on $dz$ to conclude that $\|dz\|_{L^{1,\infty}(\mathcal{M};l^2_R)} \leq 2\sqrt{2}C\|x\|_1$. ∎

Case D: *The general case.* Let $x = (x_n)_{n=1}^\infty$ be a $L^1$-bounded martingale. Write $x_n = \mathrm{Re}(x_n) + i\,\mathrm{Im}(x_n)$. Clearly, $(\mathrm{Re}(x_n))_{n=1}^\infty$ and $(\mathrm{Im}(x_n))_{n=1}^\infty$ are self-adjoint $L^1$-bounded martingales. We can decompose the self-adjoint martingales

$(\mathrm{Re}(x_n))_{n=1}^{\infty}$ and $(\mathrm{Im}(x_n))_{n=1}^{\infty}$ as in Case C. Details would be similar to the reduction of the self-adjoint case to the positive case and are left to the interested reader. The proof for the finite case is complete. ∎

Now, we will outline the adjustment for the proof to work in the semi-finite case. We will only consider the case where $x = (x_n)_{n=1}^{\infty}$ is a positive martingale and $\|x\|_1 = 1$. We remark first that in the preceding proof, the fact that $\mathcal{M}$ is a finite von Neumann algebra was used only to settle the case where $0 < \lambda < 1$. The same construction as above would apply to the semi-finite case but we were able to verify inequality (2.3) only for $\lambda \geq 1$. The only obstruction for getting inequality (2.3) for general $\lambda$ is the index $j = 0$ in the definition of $y = (y_n)_{n=1}^{\infty}$. Indeed if for $n \geq 1$, we set

$$d\gamma_n := \sum_{j=1}^{\infty} \sum_{i \leq j} p_{i,n} dx_n p_{j,n}, \tag{2.8}$$

then $dy_n = p_{0,n} dx_n p_{0,n} + d\gamma_n$ and we have the following lemma:

Lemma 2.7: $\|d\gamma\|_{L^{1,\infty}(\mathcal{M};l_C^2)} + \|dz\|_{L^{1,\infty}(\mathcal{M};l_R^2)} \leq C\|x\|_1$.

*Proof:* Note that for each $n \geq 1$, $|d\gamma_n|$ is supported by the projection $(\mathbf{1} - p_{0,n})$. As $p_0 \leq p_{0,n}$ for every $n \geq 1$, it is clear that $S_C(\gamma)$ is supported by $(\mathbf{1} - p_0)$ and, since $\tau(\mathbf{1} - p_0) \leq 1$, we can deduce that for $0 < \lambda \leq 1$, $\lambda\tau(\chi_{(\lambda,\infty)}(S_C(\gamma))) \leq 1$. The case $1 < \lambda$ is done exactly as in the finite case. The same observation applies to $dz$. ∎

In light of the preceding lemma, it is enough to consider the "right" decomposition of the sequence $(p_{0,n} dx_n p_{0,n})_{n=1}^{\infty}$. To this end, as in Proposition 1.4, we will decompose $p_{0,n}$ into pairwise disjoint sequence of projections. For $n \geq 1$ and $i \geq 0$, we set

$$e_{i,n} := \bigwedge_{k=0}^{i} (q_n^{(2^{-k})} \wedge p_{0,n}) - \bigwedge_{k=0}^{i+1} (q_n^{(2^{-k})} \wedge p_{0,n}). \tag{2.9}$$

Similarly,

$$e_i := \bigwedge_{k=0}^{i} (q^{(2^{-k})} \wedge p_0) - \bigwedge_{k=0}^{i+1} (q^{(2^{-k})} \wedge p_0).$$

*Remarks 2.8:* We have the following immediate properties:

(i) For each $n \geq 1$, $(e_{i,n})_{i=0}^{\infty}$ is a sequence of disjoint projections.

(ii) For every $m \geq 1$, $\sum_{i=0}^{m} e_{i,n} = p_{0,n} - \bigwedge_{k=0}^{m+1} (q_n^{(2^{-k})} \wedge p_{0,n})$.

(iii) For every $m \geq 1$, $\sum_{i=m}^{\infty} e_{i,n} = \bigwedge_{k=0}^{m+1}(q_n^{(2^{-k})} \wedge p_{0,n}) - \bigwedge_{k=0}^{\infty}(q_n^{(2^{-k})} \wedge p_{0,n})$. In particular, $\sum_{k=m}^{\infty} e_{i,n} \leq q_n^{(2^{-m})}$.

(iv) For every $n \geq 1$, $\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} e_{i,n} dx_n e_{j,n} = p_{0,n} dx_n p_{0,n}$.

We are now ready to provide the decomposition of $(p_{0,n}dx_np_{0,n})_{n=1}^{\infty}$ in the same fashion as above. For every $n \geq 1$,

$$\begin{cases} d\Xi_n := \sum_{j=0}^{\infty}\sum_{i\leq j} e_{i,n}dx_ne_{j,n} & \text{and} \\ d\Psi_n := \sum_{j=0}^{\infty}\sum_{i>j} e_{i,n}dx_ne_{j,n}. \end{cases} \tag{2.10}$$

Clearly, $p_{0,n}dx_np_{0,n} = d\Xi_n + d\Psi_n$ for every $n \geq 1$. As above, $d\Xi_n$ and $d\Psi_n$ belong to $L^{1,\infty}(\mathcal{M}_n, \tau_n)$ and we claim that there is a numerical constant $C$ with

$$\|d\Psi\|_{L^{1,\infty}(\mathcal{M};l_C^2)} + \|d\Xi\|_{L^{1,\infty}(\mathcal{M};l_R^2)} \leq C. \tag{2.11}$$

If this is verified, then it is enough to set, for every $n \geq 1$, $x_n = (\Psi_n + \gamma_n) + (\Xi_n + z_n)$. As noted in the proof of the finite case, it is enough to verify the inequality for $S_C(\Psi)$, that is, we need to show that for every $0 < \lambda < \infty$,

$$\lambda\tau(\chi_{(\lambda,\infty)}(S_C(\Psi))) \leq C.$$

For $\lambda \geq 1$, the preceding inequality can be deduced as follows: first, it is clear that $\tau(\chi_{(\lambda,\infty)}(S_C(\Psi))) \leq \lambda^{-2}\|S_C(\Psi)\|_2^2$. Note that since the triangular truncation is a contractive projection in $L^2(\mathcal{M},\tau)$, we have

$$\begin{aligned} \|S_C(\Psi)\|_2^2 &= \sum_{n=1}^{\infty} \|d\Psi_n\|_2^2 \\ &\leq \sum_{n=1}^{\infty} \|p_{0,n}dx_np_{0,n}\|_2^2 \\ &\leq \sum_{n=1}^{\infty} \|q_n^{(1)}dx_nq_n^{(1)}\|_2^2 \\ &\leq \sum_{n=1}^{\infty} \|q_n^{(1)}x_nq_n^{(1)} - q_{n-1}^{(1)}x_{n-1}q_{n-1}^{(1)}\|_2^2. \end{aligned}$$

Using Lemma 2.5 and Lemma 2.6 for $n_0 = 0$, we can conclude that $\|S_C(\Psi)\|_2^2 \leq 24$ and therefore $\tau(\chi_{(\lambda,\infty)}(S_C(\Psi))) \leq 24\lambda^{-2}$. Hence, for $\lambda \geq 1$, we get that $\lambda\tau(\chi_{(\lambda,\infty)}(S_C(\Psi))) \leq 24$.

For $\lambda < 1$, we will consider the case $\lambda = 2^{-n_0}$ for some $n_0 > 1$. The proof follows the same pattern as in the finite case. First, we make a reduction as in Lemma 2.3:

LEMMA 2.9: *Let* $f_{n_0} = \sum_{i=n_0}^{\infty} e_i$, $\alpha \in (0,1)$ *and* $\beta \in (0,1)$. *Then:*

(i) $\tau(\mathbf{1} - f_{n_0}) \leq 2^{n_0+1}$;

(ii) $\tau(\chi_{(2^{-n_0},\infty)}(S_C(\Psi))) \leq \alpha^{-1}\beta^{-1}4^{n_0}\tau(f_{n_0}S_C(\Psi)^2 f_{n_0}) + 4(1-\alpha)^{-1}2^{n_0}$.

*Proof:* For the first part, note that $\mathbf{1} - f_{n_0} = \sum_{i=0}^{n_0-1} e_i = p_0 - (\bigwedge_{k=0}^{n_0} q^{(2^{-k})}) \wedge p_0$. By the Kaplansky formula (see, for instance, [KR1, Theorem 6.1.6, p. 403]), we have the equivalence, $p_0 - (\bigwedge_{k=0}^{n_0} q^{(2^{-k})}) \wedge p_0 \sim p_0 \vee (\bigwedge_{k=0}^{n_0} q^{(2^{-k})}) - \bigwedge_{k=0}^{n_0} q^{(2^{-k})}$, so $\mathbf{1} - f_{n_0} \precsim \mathbf{1} - \bigwedge_{k=0}^{n_0} q^{(2^{-k})}$. We can estimate the trace as follows:

$$\begin{aligned}\tau(\mathbf{1} - f_{n_0}) &\leq \tau(\mathbf{1} - \bigwedge_{k=0}^{n_0} q^{(2^{-k})})\\ &= \tau(\bigvee_{k=0}^{n_0}(\mathbf{1} - q^{(2^{-k})}))\\ &\leq \sum_{k=0}^{n_0} \tau(\mathbf{1} - q^{(2^{-k})})\\ &\leq \sum_{k=0}^{n_0} 2^k \leq 2^{n_0+1}.\end{aligned}$$

This proves the first part. The second part is done exactly as in Lemma 2.3. ∎

We also have the corresponding result to Lemma 2.4:

LEMMA 2.10: *The sequence* $(q_n^{(2^{-n_0})} x_n q_n^{(2^{-n_0})})_{n=1}^{\infty}$ *is also a supermartingale and* $\tau(f_{n_0}S_C(\Psi)^2 f_{n_0}) \leq \sum_{n=1}^{\infty} \|q_n^{(2^{-n_0})} x_n q_n^{(2^{-n_0})} - q_{n-1}^{(2^{-n_0})} x_{n-1} q_{n-1}^{(2^{-n_0})}\|_2^2$.

*Proof:* As in the proof of Lemma 2.4, we have, for every $N \geq 1$,

$$f_{n_0} S_{C,N}^2(\Psi) f_{n_0} = \sum_{n=1}^{N}\sum_{l=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i\geq max(l,j)} f_{n_0} e_{l,n} dx_n e_{i,n} dx_n e_{j,n} f_{n_0}.$$

We remark that if $l < n_0$, then $f_{n_0} e_{l,n} = e_{l,n} f_{n_0} = 0$. For this, we note that as $e_{l,n} = \bigwedge_{k=0}^{l}(q_n^{(2^{-k})} \wedge p_{0,n}) - \bigwedge_{k=0}^{l+1}(q_n^{(2^{-k})} \wedge p_{0,n})$, $q^{(2^{-k})} \leq q_n^{(2^{-k})}$ for all $k \geq 1$ and $p_0 \leq p_{0,n}$, it is clear that $f_{n_0} = \bigwedge_{k=0}^{n_0+1}(q^{(2^{-k})} \wedge p_0) - \bigwedge_{k=0}^{\infty}(q^{(2^{-k})} \wedge p_0)$ is a subprojection of $\bigwedge_{k=0}^{l+1}(q_n^{(2^{-k})} \wedge p_{0,n})$ when $l < n_0$ and, by the definition of $e_{l,n}$, it follows that $f_{n_0} \perp e_{l,n}$. With this property, we can rewrite the above equality as

$$\begin{aligned} f_{n_0} S^2_{C,N}(\Psi) f_{n_0} &= \sum_{n=1}^{N} \sum_{l=n_0}^{\infty} \sum_{j=n_0}^{\infty} \sum_{i\geq \max(l,j)} f_{n_0} e_{l,n} dx_n e_{i,n} dx_n e_{j,n} f_{n_0} \\ &= \sum_{n=1}^{N} f_{n_0} \Big( \sum_{l=n_0}^{\infty} \sum_{j=n_0}^{\infty} \sum_{i\geq \max(l,j)} e_{l,n} dx_n e_{i,n} dx_n e_{j,n} \Big) f_{n_0}. \end{aligned}$$

Taking the trace, we get

$$\begin{aligned} \tau(f_{n_0} S^2_{C,N}(\Psi) f_{n_0}) &\leq \sum_{n=1}^{N} \sum_{l=n_0}^{\infty} \sum_{j=n_0}^{\infty} \sum_{i\geq \max(l,j)} \tau(e_{l,n} dx_n e_{i,n} dx_n e_{j,n}) \\ &= \sum_{n=1}^{N} \sum_{l=n_0}^{\infty} \tau\Big(e_{l,n} dx_n \Big( \sum_{i\geq l} e_{i,n} \Big) dx_n e_{l,n}\Big) \\ &\leq \sum_{n=1}^{N} \sum_{l=n_0}^{\infty} \tau\Big(e_{l,n} dx_n \Big( \sum_{i=n_0}^{\infty} e_{i,n} \Big) dx_n e_{l,n}\Big). \end{aligned}$$

From Remark 2.8(iii), $\sum_{i=n_0}^{\infty} e_{i,n} \leq q_n^{(2^{-n_0})}$, and therefore

$$\begin{aligned} \tau(f_{n_0} S^2_{C,N}(\Psi) f_{n_0}) &\leq \sum_{n=1}^{N} \sum_{l=n_0}^{\infty} \tau(e_{l,n} dx_n q_n^{(2^{-n_0})} dx_n e_{l,n}) \\ &= \sum_{n=1}^{N} \tau\Big( \Big( \sum_{l=n_0}^{\infty} e_{l,n} \Big) dx_n q_n^{(2^{-n_0})} dx_n \Big) \\ &= \sum_{n=1}^{N} \tau\Big( q_n^{(2^{-n_0})} dx_n \Big( \sum_{l=n_0}^{\infty} e_{l,n} \Big) dx_n q_n^{(2^{-n_0})} \Big) \\ &\leq \sum_{n=1}^{N} \tau(q_n^{(2^{-n_0})} dx_n q_n^{(2^{-n_0})} dx_n q_n^{(2^{-n_0})}). \end{aligned}$$

We can deduce the stated inequality as in the proof of Lemma 2.4. ∎

Once we have the inequality in the preceding lemma, the rest of the proof is accomplished exactly as in the finite case with $2^{-n_0}$ instead of $2^{n_0}$. ∎

Theorem 2.1 can be extended to square functions of non-commutative submartingales and non-commutative supermartingales.

Corollary 2.11: *There exists a constant $K$ such that if $s = (s_n)_{n=1}^{\infty}$ is either a submartingale or a supermartingale and is bounded in $L^1(\mathcal{M}, \tau)$, then there exist two sequences $y = (y_n)_{n=1}^{\infty}$ and $w = (w_n)_{n=1}^{\infty}$ such that:*

(i) *for every* $n \geq 1$, $s_n = y_n + w_n$;
(ii) *for every* $n \geq 1$, $y_n$ *and* $w_n$ *belong to* $L^{1,\infty}(\mathcal{M}_n, \tau_n)$;
(iii) $\|dy\|_{L^{1,\infty}(\mathcal{M};l^2_C)} + \|dw\|_{L^{1,\infty}(\mathcal{M};l^2_R)} \leq K \sup_{n\geq 1} \|x_n\|_1$.

For the proof, we will use the following lemma whose proof can be found in [A, Proposition 2.4]:

LEMMA 2.12: *Let* $(a_i)_{i=1}^m$ *be a finite sequence in* $\mathcal{M}$, $w = (\sum_{i=1}^m |a_i|^2)^{1/2}$ *and* $s(w)$ *is the support projection of* $w$. *Then there exists a unique sequence* $(b_i)_{i=1}^m$ *so that:*
(1) *for every* $1 \leq j \leq m$, $b_j w = a_j$;
(2) $\sum_{i=1}^m b_i^* b_i = s(w)$;
(3) $\sum_{i=1}^m b_i^* a_i = w$.
*In particular,* $\|(\sum_{i=1}^m |a_i|^2)^{1/2}\|_1 \leq \sum_{i=1}^m \|a_i\|_1$.

*Proof of Corollary 2.11:* We will present the proof for the case of a submartingale (the adjustment to the supermartingale case is straightforward). The reduction to the case of Theorem 2.1 is done by splitting the submartingale $(s_n)_{n=1}^\infty$ into the sum of a martingale and an increasing sequence of positive operators. As before, let

$$x_n := \begin{cases} s_1 & \text{for } n = 1, \\ s_n + \sum_{l=1}^{n-1} s_l - \mathcal{E}_l(s_{l+1}) & \text{for } n \geq 2, \end{cases}$$

and

$$v_n := \begin{cases} 0 & \text{for } n = 1, \\ \sum_{l=1}^{n-1} \mathcal{E}_l(s_{l+1}) - s_l & \text{for } n \geq 2. \end{cases}$$

The following properties are immediate:
(a) $(x_n)_{n=1}^\infty$ is a martingale with $\|x_n\|_1 \leq \|s_n\|_1$;
(b) for every $n \geq 1$, $x_n + v_n = s_n$;
(c) for every $n \geq 2$, $v_n \geq v_{n-1} \geq \cdots \geq v_1 = 0$.

Moreover, for every $n \geq 1$,

$$\begin{aligned} \|v_n\|_1 &= \tau(v_n) \\ &= \sum_{l=1}^{n-1} \tau(\mathcal{E}_l(s_{l+1}) - s_l) \\ &= \sum_{l=1}^{n-1} \tau(s_{l+1} - s_l) \\ &= \tau(s_{n-1} - s_1) \leq 2\|s_n\|_1, \end{aligned}$$

so $(v_n)_{n=1}^\infty$ is an increasing, $L^1$-bounded sequence of positive operators.

Applying Theorem 2.1 to the martingale $x = (x_n)_{n=1}^{\infty}$, we have the decomposition $x_n = y_n + z_n$. Moreover, from Lemma 2.12, we can conclude that

$$\begin{aligned}\|dv\|_{L^1(\mathcal{M};l_R^2)} &\leq \sum_{n=1}^{\infty} \|dv_n\|_1 \\ &= \sup_{m\geq 1} \sum_{n=1}^{m} \|dv_n\|_1 \\ &= \sup_{m\geq 1} \|v_m\|_1 \leq 2 \sup_{m\geq 1} \|s_m\|_1.\end{aligned}$$

For $n \geq 1$, set

$$w_n = z_n + v_n.$$

It is clear that $s_n = y_n + w_n$ for every $n \geq 1$. All required conditions follow from properties of $(y_n)_{n=1}^{\infty}$ and $(w_n)_{n=1}^{\infty}$. Details are left to the reader. ∎

Assume that $\mathcal{M}$ is finite and $\tau$ is a normalized trace. From the fact that $\|\cdot\|_p \leq \|\cdot\|_{1,\infty}$ for every $0 < p < 1$, we can also state:

COROLLARY 2.13: *Assume that $\mathcal{M}$ is finite and $\tau$ is a normalized trace. For every $0 < p < 1$, there exists a constant $\kappa_p$ (depending only on $p$) such that if $s = (s_n)_{n=1}^{\infty}$ is either a submartingale or a supermartingale and is bounded in $L^1(\mathcal{M}, \tau)$, then there exist two sequences $y = (y_n)_{n=1}^{\infty}$ and $w = (w_n)_{n=1}^{\infty}$ such that:*

(i) *for every $n \geq 1$, $s_n = y_n + w_n$;*
(ii) *for every $n \geq 1$, $y_n$ and $w_n$ belong to $L^p(\mathcal{M}_n, \tau_n)$;*
(iii) $\|dy\|_{L^p(\mathcal{M};l_C^2)} + \|dw\|_{L^p(\mathcal{M};l_R^2)} \leq \kappa_p \sup_{n\geq 1} \|x_n\|_1$.

It would be desirable to have the decomposition in Theorem 2.1 to be in $L^1(\mathcal{M}_n, \tau_n)$ and the sequences $y = (y_n)_{n=1}^{\infty}$ and $z = (z_n)_{n=1}^{\infty}$ being martingales in $L^1(\mathcal{M}, \tau)$. Of course, in the hyperfinite case, the decomposition can be chosen to be in $L^1(\mathcal{M}_n, \tau_n)$ since $L^1(\mathcal{M}_n, \tau_n)$'s are finite dimensional, but for the general case this is still open. We state this question explicitly.

PROBLEM 2.14: *Does there exist an absolute constant $C$ such that if $x = (x_n)_{n=1}^{\infty}$ is a martingale in $L^1(\mathcal{M}, \tau)$, then there exist two $L^1$-martingales $y = (y_n)_{n=1}^{\infty}$ and $z = (z_n)_{n=1}^{\infty}$ such that $x_n = y_n + z_n$ for every $n \geq 1$ and $\|S_C(y)\|_{1,\infty} + \|S_R(z)\|_{1,\infty} \leq C\|x\|_1$?*

It is also unclear if our approach can be used to settle the case of conditioned square functions using our method. We will state this question explicitly: Let

$x = (x_n)_{n\geq 1}$ be a finite sequence in $\mathcal{M}$ (not necessarily a martingale). Following [JX1], we consider the quantities

$$\sigma_C(x) = \Big(\sum_{n\geq 1} \mathcal{E}_{n-1}(|dx_n|^2)\Big)^{1/2} \quad \text{and} \quad \sigma_R(x) = \Big(\sum_{n\geq 1} \mathcal{E}_{n-1}(|dx_n^*|^2)\Big)^{1/2}$$

with the convention that $\mathcal{E}_0 = \mathcal{E}_1$. These are the conditioned square functions of the sequence $(x_n)_{n\geq 1}$. Note that since $(x_n)_{n\geq 1}$ is a sequence in $\mathcal{M}$, $|dx_n|^2 \in \mathcal{M}$ for every $n \geq 1$ and therefore $\mathcal{E}_{n-1}(|dx_n|^2)$ and $\mathcal{E}_{n-1}(|dx_n^*|^2)$ are well-defined for every $n \geq 1$. In fact, one needs to consider sequences in $L^p(\mathcal{M}, \tau)$ for $2 \leq p \leq \infty$. Generalizations of the non-commutative Burkholder inequalities were considered by Junge and Xu in [JX1, Theorem 6.1] for the case $p > 1$. We were also unable to settle the case $p = 1$ of conditioned square functions using our method. We will state this question explicitly:

PROBLEM 2.15: *Does there exist an absolute constant $K$ such that if $x = (x_n)_{n=1}^\infty$ is a finite martingale in $\mathcal{M}$, then there exist two sequences $y = (y_n)_{n=1}^\infty$ and $z = (z_n)_{n=1}^\infty$ such that $x_n = y_n + z_n$ for every $n \geq 1$ and $\|\sigma_C(y)\|_{1,\infty} + \|\sigma_R(z)\|_{1,\infty} \leq K\|x\|_1$?*

## 3. Consequences to Hardy spaces

Throughout this section, we assume that $\mathcal{M}$ is finite and the trace $\tau$ is normalized. We recall the definitions of martingale Hardy spaces and martingale $BMO$. For $1 \leq p < \infty$, $\mathcal{H}_C^p(\mathcal{M})$ (respectively $\mathcal{H}_R^p(\mathcal{M})$) is defined as the set of all $L^p$-martingales $x$ with respect to a filtration $(\mathcal{M}_n)_{n\geq 1}$ such that $dx \in L^p(\mathcal{M}; l_C^2)$ (respectively $L^p(\mathcal{M}; l_R^2)$), and set

$$\|x\|_{\mathcal{H}_C^p(\mathcal{M})} = \|dx\|_{L^p(\mathcal{M}; l_C^2)} \quad \text{and} \quad \|x\|_{\mathcal{H}_R^p(\mathcal{M})} = \|dx\|_{L^p(\mathcal{M}; l_R^2)}.$$

Equipped with the previous norms, $\mathcal{H}_C^p(\mathcal{M})$ and $\mathcal{H}_R^p(\mathcal{M})$ are Banach spaces. The Hardy space of non-commutative martingale is defined as follows: if $1 \leq p < 2$,

$$\mathcal{H}^p(\mathcal{M}) = \mathcal{H}_C^p(\mathcal{M}) + \mathcal{H}_R^p(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}^p(\mathcal{M})} = \inf\{\|y\|_{\mathcal{H}_C^p(\mathcal{M})} + \|z\|_{\mathcal{H}_R^p(\mathcal{M})}\}$$

where the infimum runs over all $y \in \mathcal{H}_C^p(\mathcal{M})$ and $z \in \mathcal{H}_R^p(\mathcal{M})$ that satisfy $x = y + z$; and if $2 \leq p < \infty$,

$$\mathcal{H}^p(\mathcal{M}) = \mathcal{H}_C^p(\mathcal{M}) \cap \mathcal{H}_R^p(\mathcal{M})$$

equipped with the norm

$$\|x\|_{\mathcal{H}^p(\mathcal{M})} = \max\{\|x\|_{\mathcal{H}^p_C(\mathcal{M})}, \ \|x\|_{\mathcal{H}^p_R(\mathcal{M})}\}.$$

The martingale $BMO$ was defined in [PX] as follows:

$$BMO_C(\mathcal{M}) = \{a \in L^2(\mathcal{M},\tau)\colon \sup_{n\geq 1} \|\mathcal{E}_n|a - \mathcal{E}_{n-1}a|^2\|_\infty < \infty\}$$

with $\mathcal{E}_0 a = 0$. The space $BMO_C(\mathcal{M})$ is a Banach space when equipped with the norm

$$\|a\|_{BMO_C(\mathcal{M})} = \sup_{n\geq 1} \|\mathcal{E}_n|a - \mathcal{E}_{n-1}a|^2\|_\infty^{1/2}.$$

The space $BMO_R(\mathcal{M})$ is defined as the space of all $a$ such that $a^* \in BMO_C(\mathcal{M})$ with $\|a\|_{BMO_R(\mathcal{M})} = \|a^*\|_{BMO_C(\mathcal{M})}$. The space $BMO(\mathcal{M})$ is the intersection of $BMO_C(\mathcal{M})$ and $BMO_R(\mathcal{M})$ with

$$\|a\|_{BMO(\mathcal{M})} = \max\left\{\|a\|_{BMO_C(\mathcal{M})}, \|a\|_{BMO_R(\mathcal{M})}\right\}.$$

It was established in [PX] that, as in the classical case, the dual of $\mathcal{H}^1(\mathcal{M})$ is $BMO(\mathcal{M})$.

Recall also the Zygmund space $L\log L$ and its dual $L_{exp}$. If $L^0(\Omega,\mathcal{F},P)$ is the space of all (classes) of measurable functions on a given probability space $(\Omega,\mathcal{F},P)$, the class $L\log L$ is defined by setting

$$L\log L = \left\{f \in L^0(\Omega,\mathcal{F},P)\colon \int |f|\log^+|f| \, dP < \infty\right\}.$$

Set $\|f\|_{L\log L} = \int |f|\log^+|f| \, dP$. Note that $\|\cdot\|_{L\log L}$ is not a norm but is equivalent to the norm $\|f\| = \int_0^1 f^*(t)\log(1/t)\,dt$. As a Banach space, the dual of $L\log L$ consists of the class of functions

$$L_{exp} = \left\{f \in L^0(\Omega,\mathcal{F},P)\colon \sup_{0<t<1} \frac{f^{**}(t)}{1+\log(1/t)} < \infty\right\}$$

with norm $\|f\|_{L_{exp}} = \sup_{0<t<1} f^{**}(t)(1+\log(1/t))^{-1}$, where $f^*$ is the usual decreasing rearrangement of $f$ and $f^{**}(t) = \int_0^t f^*(s)\,ds$.

The spaces $L\log L$ and $L_{exp}$ are rearrangement invariant Banach function spaces (see for instance [BS, Theorem 6.4, pp. 246–247]) so non-commutative analogues $L\log L(\mathcal{M},\tau)$ and $L_{exp}(\mathcal{M},\tau)$ respectively are well defined as described in Section 2. We remark that if a martingale $x$ is bounded in $L\log L(\mathcal{M},\tau)$, then it is uniformly integrable in $L^1(\mathcal{M},\tau)$ and therefore is of the form $x = (\mathcal{E}_n(x_\infty))_{n=1}^\infty$ with $x_\infty \in L\log L(\mathcal{M},\tau)$.

The next theorem is the principal result of this section. It improves on a result from [R1] and generalizes to the non-commutative case a classical inequality (see, for instance, [G, Theorem III 3.2].

THEOREM 3.1: *There is a constant $K$ such that if $x = (x_n)_{n=1}^{\infty}$ is a martingale that is bounded in $L\log L(\mathcal{M},\tau)$. Then*

$$\|x\|_{\mathcal{H}^1(\mathcal{M})} \leq K + K\|x_\infty\|_{L\log L(\mathcal{M},\tau)}.$$

By duality, we immediately obtain the following:

COROLLARY 3.2: *There is a constant $C$ such that for every $x \in BMO(\mathcal{M})$,*

$$\|x\|_{L_{exp}(\mathcal{M},\tau)} \leq C\|x\|_{BMO(\mathcal{M})}.$$

The proof of the theorem is based on the following consequence of Theorem 2.1. Recall the Lorentz-space $L^{p,1}(\mathcal{M},\tau)$ as the space of all $a \in \overline{\mathcal{M}}$ for which

$$\|a\|_{p,1} = \int_0^1 \mu_t(a)t^{1/p}\,\frac{dt}{t} < \infty.$$

Equipped with such a norm, $L^{p,1}(\mathcal{M},\tau)$ is a Banach space.

PROPOSITION 3.3: *Let $1 < p < 2$. For any $L^p$-bounded martingale $x = (x_n)_{n=1}^{\infty}$,*

$$\|x\|_{\mathcal{H}^p(\mathcal{M})} \leq \gamma_p\|x_\infty\|_{p,1}$$

*where $\gamma_p \leq C(p-1)^{-1}$ for some absolute constant $C$.*

*Proof:* It is enough to assume that $(x_n)_{n=1}^{\infty}$ be a positive $L^2$-bounded martingale. By Theorem 2.1, there exists a decomposition $(y_n)_{n=1}^{\infty}$ and $(z_n)_{n=1}^{\infty}$ such that

$$\|dy\|_{L^{1,\infty}(\mathcal{M};l^2_C)} + \|dz\|_{L^{1,\infty}(\mathcal{M};l^2_R)} \leq K\|x\|_1.$$

The construction of $dy$ and $dz$ in Equation (2.1) also reveals that the same decomposition satisfies

$$\|dy\|_{L^2(\mathcal{M};l^2_C)} + \|dz\|_{L^2(\mathcal{M};l^2_R)} \leq \|x\|_2.$$

We can deduce that $\|dy\|_{L^p(\mathcal{M};l^2_C)} + \|dz\|_{L^p(\mathcal{M};l^2_R)} \leq C\|x\|_1^{\theta}\|x\|_2^{1-\theta}$ for an absolute constant $C$ and an appropriate $0 < \theta < 1$, so we can conclude that

$$\|dx\|_{L^p(\mathcal{M};l^2_C)+L^p(\mathcal{M};l^2_R)} \leq C\|x\|_{p,1}.$$

Applying the non-commutative Stein's inequality, we obtain that

$$\|x\|_{\mathcal{H}^p(\mathcal{M})} \leq C\gamma_p\|x\|_{p,1}$$

where $\gamma_p$ is of order $(p-1)^{-1}$ as $p \to 1$ (see for instance [R1, Theorem 5.3], also [JX2] for more in-depth discussion of order of growth of different martingale inequalities). ∎

*Proof of Theorem 3.1:* The proof presented below is reminiscent of an old argument used in the book [Z, Vol II, p. 119] for the classical Hilbert transform. Let $x = (\mathcal{E}_n(x_\infty))_{n=1}^\infty$ be a martingale with $\tau(|x_\infty|\log^+|x_\infty|) < \infty$. Let $a = |x_\infty|$ and set $(e_t)_t$ to be the spectral decomposition of $a$. For each $k \in \mathbb{N}$, let $P_k = \chi_{[2^{k-1},2^k)}(a)$ be the spectral projection relative to $[2^{k-1}, 2^k)$. Define $a_k = aP_k$ for $k \geq 1$ and $a_0 = a\chi_{[0,1)}(a)$. Clearly $a = \sum_{k=0}^\infty a_k$ in $L^1(\mathcal{M}, \tau)$.

For every $k \in \mathbb{N}$, consider the martingale $x^{(k)} = (\mathcal{E}_n(x_\infty P_k))_{n=1}^\infty$. Then from Proposition 3.3,

$$\|x^{(k)}\|_{\mathcal{H}^1(\mathcal{M})} \leq \|x^{(k)}\|_{\mathcal{H}^p(\mathcal{M})} \leq \gamma_p\|x^{(k)}\|_{p,1}.$$

So for every $1 < p < 2$, there is a constant $C$ such that

$$\|x^{(k)}\|_{\mathcal{H}^1(\mathcal{M})} \leq C(p-1)^{-1}\|x^{(k)}\|_{p,1}.$$

Since $\|x^{(k)}\|_{p,1} = \|a_k\|_{p,1}$ and $a_k \leq 2^k P_k$, we get, for $1 < p < 2$,

$$\|x^{(k)}\|_{\mathcal{H}^1(\mathcal{M})} \leq C(p-1)^{-1}2^k\|P_k\|_{p,1}.$$

But we remark that $\mu_t(P_k) = \chi_{(0,\tau(P_k))}(t)$ so

$$\|P_k\|_{p,1} = \int_0^{\tau(P_k)} t^{1/p-1}\, dt = p\tau(P_k)^{1/p} \leq 2\tau(P_k)^{1/p}$$

and therefore

$$\|x^{(k)}\|_{\mathcal{H}^1(\mathcal{M})} \leq 2C(p-1)^{-1}2^k\tau(P_k)^{1/p}.$$

If we set $p = 1 + 1/(k+1)$ and $\eta_k = \tau(P_k)$, we have

$$\|x^{(k)}\|_{\mathcal{H}^1(\mathcal{M})} \leq 2C(k+1)2^k\eta_k^{(k+1)/(k+2)}.$$

Taking the summation over $k$,

$$\|x\|_{\mathcal{H}^1(\mathcal{M})} \leq \sum_{k=0}^\infty 2C(k+1)2^k\eta_k^{(k+1)/(k+2)}.$$

We note as in [Z] that if $J = \{k \in \mathbb{N};\ \eta_k \le 3^{-k}\}$ then

$$\sum_{k\in J} 2C(k+1)2^k \eta_k^{(k+1)/(k+2)} \le \sum_{k=0}^{\infty} 2C(k+1)2^k (3^{-k})^{(k+1)/(k+2)} = \alpha < \infty.$$

On the other hand, for $k \in \mathbb{N} \setminus J$, $\eta_k^{(k+1)/(k+2)} \le \eta_k 3^{k/(k+2)} \le \beta\eta_k$ where $\beta = \sup_k 3^{k/(k+2)}$. So we get

$$\begin{aligned}\|x\|_{\mathcal{H}^1(\mathcal{M})} &\le \alpha + 2C\beta \sum_{k=0}^{\infty} (k+1)2^k \eta_k \\ &\le \alpha + 2C\beta(\eta_0 + 4\eta_1) + 2C\beta \sum_{k\ge 2} (k+1)2^k \eta_k.\end{aligned}$$

Since for $k \ge 2$, $k+1 \le 3(k-1)$, we get

$$\|x\|_{\mathcal{H}^1(\mathcal{M})} \le \alpha + 10C\beta + 6C\beta \sum_{k\ge 2} (k-1)2^{k-1}\eta_k.$$

To complete the proof, notice that for $k \ge 2$,

$$\begin{aligned}(k-1)2^{k-1}\eta_k &= \int_{2^{k-1}}^{2^k} (k-1)2^{k-1}\, d\tau(e_t) \\ &\le \int_{2^{k-1}}^{2^k} \frac{t\log t}{\log 2}\, d\tau(e_t),\end{aligned}$$

as $2^{k-1} \le t$ and therefore $(k-1)\log 2 \le \log t$. Hence if we set

$$K = \max\{\alpha + 10C\beta, 6C\beta(\log 2)^{-1}\},$$

then we get

$$\|x\|_{\mathcal{H}^1(\mathcal{M})} \le K + K\tau(a\log^+ a).$$

The proof is complete. ∎

We conclude with the following remark. Recall the non-commutative analogue of Burkholder–Gundy inequalities proved in [PX]: If $1 < p < \infty$ and $x \in L^p(\mathcal{M}, \tau)$ then

$$(BG_p) \qquad \alpha_p^{-1}\|x\|_{\mathcal{H}^p(\mathcal{M})} \le \|x\|_p \le \beta_p \|x\|_{\mathcal{H}^p(\mathcal{M})}.$$

The optimal order of magnitude of the constant $\alpha_p$ (when $p \to 1$) is still an open question. It is now known that it lies between $(p-1)^{-1}$ and $(p-2)^{-2}$ (see [JX2] for more details). Theorem 3.1 provides an indication that it should be $(p-1)^{-1}$ but we are unable to verify this at this point.

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